

GEODESICS IN CONICAL MANIFOLDS

MARCO GHIMENTI

Dipartimento di Matematica, Università di Pisa,

v. Buonarroti 2, 56100, Pisa, Italy

Dipartimento di Matematica Applicata, Università di Pisa,

via Bonanno 25b, 56100, Pisa, Italy

Abstract

The aim of this paper is to extend the definition of geodesics to conical manifolds, defined as submanifolds of \mathbb{R}^n with a finite number of singularities. We look for an approach suitable both for the local geodesic problem and for the calculus of variation in the large. We give a definition which links the local solutions of the Cauchy problem (1) with variational geodesics, i.e. critical points of the energy functional. We prove a deformation lemma (Theorem 2) which leads us to extend the Lusternik-Schnirelmann theory to conical manifolds, and to estimate the number of geodesics (Theorem 10 and Corollary 10.1). In section 4, we provide some applications in which conical manifolds arise naturally: in particular, we focus on the brachistochrone problem for a frictionless particle moving in S^n or in \mathbb{R}^n in the presence of a potential $U(x)$ unbounded from below. We conclude with an appendix in which the main results are presented in a general framework.

1 Introduction and basic definition

The existence of geodesic is one of most studied problems in the calculus of variation. In this paper we want to study the presence of geodesics in a particular kind of manifolds, called conical manifolds, that appears in a natural way in some optimization problem (see section 4.1)

We define the following type of topological manifolds.

Definition 1. *A conical manifold M is a complete n -dimensional C^0 sub manifold of \mathbb{R}^m which is everywhere smooth, except for a finite set of points V . A point in V is called vertex.*

Usually there are two ways to introduce geodesics in a smooth manifold:

Local (Cauchy problem): a geodesic is a solution of a suitable Cauchy problem, i.e. given $p \in M$, $v \in T_p M$, we look for a curve $\gamma : [0, \varepsilon] \rightarrow M$ s.t.

$$\begin{cases} D_s \gamma' = 0; \\ \gamma(0) = p; \\ \gamma'(0) = v. \end{cases} \quad (1)$$

Global (Bolza problem): we consider the path space on M :

$$\begin{aligned} \Omega_{p,q} &:= \{ \gamma \in H^1([0, 1], M) : \gamma(0) = p, \gamma(1) = q \}; \\ \Omega_p &:= \{ \gamma \in H^1([0, 1], M) : \gamma(0) = \gamma(1) = p \}; \end{aligned}$$

a geodesic is a critical point of the energy functional defined by¹

$$E : \Omega \rightarrow \mathbb{R}$$

$$E(\gamma) = \int_0^1 |\gamma'(s)|^2 ds$$

In conical manifolds the Cauchy problem (1) is not well posed, and the solution is neither unique, nor continuously dependent from the initial data. The functional approach gives us an easy result on minimal geodesics. However, this approach is not completely useful: we can not easily define a critical point of energy different from minimum, because the energy is not a C^1 functional.

Furthermore, the usual generalization of the derivative, the weak slope, cannot be applied to our case, because it requires some conditions on the manifolds M which are not satisfied in the case of conical manifolds. The weak slope was introduced by Marco Degiovanni and Marco Marzocchi in [DM94] (see also [Deg97, CD95, CDM93]). Moreover we refer to [DM99, MM02] for a weak slope approach to geodesic problem and to [Ghi04] for a detailed comparison with our approach.

We give the following definition of geodesics, that appears to be the most suitable one for this kind of problem.

Definition 2. *A path $\gamma \in \Omega$ is a geodesic iff*

- *the set $T = T_\gamma := \{s \in (0, 1) : \gamma(s) \in V\}$ is a closed set without internal part;*
- *$D_s \gamma' = 0 \ \forall s \in [0, 1] \setminus T$;*
- *$|\gamma'|^2$ is constant as a function in L^1 .*

We note that a geodesic may not be a local minimum for the length functional, for example, we consider a Euclidean cone and a broken geodesic passing through the vertex. However, this definition allows us to prove the main theorem of this paper (see corollary 10.1)

Theorem 1. *Let M be a conical manifold, $p \in M$. Then there are at least $\text{cat } \Omega$ geodesics.*

We are relating definition 2, which is local, with the topology of the path space, which is a tool of the calculus of variation in the large; furthermore, this approach allows us to find also non minimal geodesics.

Unfortunately, it's not easy to compute $\text{cat } \Omega$ for conical manifolds. Set

$$\Omega_{p,q}^\infty := \{\gamma \in C^0([0, 1], M) : \gamma(0) = p, \gamma(1) = q\};$$

we know that, for a smooth manifold, there is an homotopy equivalence

$$\Omega_{p,q}^\infty \simeq \Omega_{p,q} \tag{2}$$

(for a proof see, for example [Kli78, Th 1.2.10]). In general this result is false for conical manifolds; we show it by an example.

¹Hereafter we simply note Ω when we not need to specify the extremal points of paths.

Example 1. Let

$$M = \left\{ \left(x, x \sin \frac{1}{x} \right), x \in \mathbb{R} \right\} \subset \mathbb{R}^2;$$

this is an 1-dimensional conical manifold with vertex $O = (0, 0)$. Let $p, q \in M$ be two opposite points with respect to O : we have that, while $\Omega_{p,q}^\infty$ is connected, $\Omega_{p,q}$ is not, so the usual homotopy equivalence 2 does not hold.

Even if an explicit calculation of $\text{cat } \Omega$ in general is very difficult, in section 4, we will give a criterion for which (2) holds. Moreover we show some applications in which conical manifolds appears naturally.

2 Deformation lemmas

We want to prove that our definition of geodesic is compatible with the energy functional, i.e. if there is no geodesic of energy c , then there is no change of the topology of functional E at level c . To do that, we prove a deformation lemma (Theorem 2), that is the main result of this section.

Definition 3. Given $p \in M$ we set

$$\begin{aligned} \Omega^b &= \Omega_p^b := \{ \gamma \in \Omega_p : E(\gamma) \leq b \}; \\ \Omega_a^b &= \Omega_{a,p}^b := \{ \gamma \in \Omega_p : a \leq E(\gamma) \leq b \}. \end{aligned}$$

Theorem 2 (Deformation lemma). *Let M be a conical manifold, $p \in M$. Suppose that there exists $c \in \mathbb{R}$ s.t. Ω^c contains only a finite number of geodesics. Then if $a, b \in \mathbb{R}$, and $a < b < c$ are s.t. the strip $[a, b]$ contains only regular values of E , Ω^a is a deformation retract of Ω^b .*

In order to prove this theorem, we must study the structure of Ω^c . For the moment, we consider a special case.

We suppose that M has only a vertex v , and we study the special closed geodesic γ_0 for which there exists a unique σ s.t. $\gamma_0(\sigma) = v$. We set $E(\gamma_0) = c_0$ and we suppose that there exist $a, b \in \mathbb{R}$, $c_0 < a < b$, s.t. Ω^b contains only the geodesics γ_0 (so Ω_a^b contains no geodesics).

At last let us set

$$L_1 = \int_0^\sigma |\gamma_0'|^2, \quad L_2 = \int_\sigma^1 |\gamma_0'|^2.$$

We identify now two special subsets of Ω_a^b . Let

$$\Sigma = \{ \gamma \in \Omega_a^b, \text{ s.t. } v \in \text{Im} \gamma \}; \quad (3)$$

for every $\gamma \in \Sigma$ it exists a set T s.t. $\gamma(s) = v$ iff $s \in T$. Let

$$\Sigma_0 = \left\{ \begin{array}{l} \gamma \in \Sigma, \text{ s.t. } D_s \gamma'(s) = 0 \text{ and } |\gamma'|^2 \text{ is constant} \\ \text{on every connected component of } [0, 1] \setminus T \end{array} \right\}. \quad (4)$$

Indeed, we will see in the proof of the next lemma that, if $\gamma \in \Sigma_0$, then $\gamma([0, 1]) = \gamma_0([0, 1])$, so Σ_0 is the set of the piecewise geodesics that are equivalent to γ_0 up to affine reparametrization.

Lemma 3. Σ_0 is compact.

Proof. If $\gamma \in \Sigma_0$, only two situations occur: either $\exists! \tau$ s.t. $\gamma(\tau) = v$, or $\exists[\tau_1, \tau_2]$ s.t. $\gamma(t) = v$ iff $t \in [\tau_1, \tau_2]$. In fact, if it were two isolated consecutive points $s_1, s_2 \in T$ s.t. $\gamma(s_i) = v$, then, we can obtain by reparametrization a geodesic $\gamma_1 \neq \gamma_0$ in Ω^b , that contradicts our assumptions (this proves also that $\gamma([0, 1]) = \gamma_0([0, 1])$).

Now take $(\gamma_n)_n \subset \Sigma_0$. For simplicity we can suppose that there exists a subsequence such that $\forall n \exists! \tau_n$ for which $\gamma_n(\tau_n) = v$ (else, definitely, $\exists[\tau_n^1, \tau_n^2]$ s.t. $\gamma_n(t) = v$ iff $s \in [\tau_n^1, \tau_n^2]$, but the proof follows in the same way).

If we consider $\|\gamma\|_{H^1} = E(\gamma)$, then we have

$$a \leq \|\gamma_n\| \leq b,$$

hence, up to subsequence, $\exists \bar{\gamma}$ s.t. $\gamma_n \rightarrow \bar{\gamma}$ in weak- H^1 norm and uniformly. Also, we know that $\forall n \exists \tau_n$ s.t. $\gamma_n(\tau_n) = v$ and

$$\gamma_n = \begin{cases} \gamma_0 \left(\frac{\sigma}{\tau_n} s \right), & s \in [0, \tau_n] \\ \gamma_0 \left(\frac{1-\sigma}{1-\tau_n} s + \frac{\sigma-\tau_n}{1-\tau_n} \right), & s \in (\tau_n, 1] \end{cases} \quad (5)$$

It exists $0 < p < 1$ such that $p \leq \tau_n \leq 1 - p$, in fact

$$\begin{aligned} b &\geq \int |\gamma_n'|^2 = \int_0^{\tau_n} |\gamma_n'|^2 + \int_{\tau_n}^1 |\gamma_n'|^2 = \\ &= \left[\frac{\sigma}{\tau_n} \right]^2 \int_0^{\tau_n} |\gamma_0'|^2 \left(\frac{\sigma}{\tau_n} s \right) + \\ &\quad + \left[\frac{1-\sigma}{1-\tau_n} \right]^2 \int_{\tau_n}^1 |\gamma_0'|^2 \left(\frac{1-\sigma}{1-\tau_n} s + \frac{\sigma-\tau_n}{1-\tau_n} \right) ds = \\ &= \frac{\sigma}{\tau_n} \int_0^\sigma |\gamma_0'|^2(s') ds' + \frac{1-\sigma}{1-\tau_n} \int_0^\sigma |\gamma_0'|^2(s') ds' = \\ &= \frac{L_1^2}{\sigma \tau_n} + \frac{L_2^2}{(1-\sigma)(1-\tau_n)}, \end{aligned}$$

so

$$b \geq \frac{L_1^2}{\sigma \tau_n} \Rightarrow \tau_n > \frac{L_1^2}{\sigma b}, \quad (6)$$

and

$$b \geq \frac{L_2^2}{(1-\sigma)(1-\tau_n)} \Rightarrow \tau_n < 1 - \frac{L_2^2}{b(1-\sigma)}. \quad (7)$$

So a subsequence exists such that $\tau_n \rightarrow \tau$, $p \leq \tau \leq 1 - p$. Obviously

$$\begin{aligned} \frac{\sigma}{\tau_n} s &\rightarrow \frac{\sigma}{\tau} s, \\ \frac{1-\sigma}{1-\tau_n} s + \frac{\sigma-\tau_n}{1-\tau_n} &\rightarrow \frac{1-\sigma}{1-\tau} s + \frac{\sigma-\tau}{1-\tau}. \end{aligned}$$

So for almost all s we have

$$\gamma_n \rightarrow \tilde{\gamma}(s) = \begin{cases} \gamma_0 \left(\frac{\sigma}{\tau} s \right), & s \in [0, \tau] \\ \gamma_0 \left(\frac{1-\sigma}{1-\tau} s + \frac{\sigma-\tau}{1-\tau} \right), & s \in (\tau, 1] \end{cases}. \quad (8)$$

Both γ_n and $\tilde{\gamma}$ are continuous, because γ_0 is continuous, so the convergence in (8) is uniform; furthermore, $\tilde{\gamma} = \tilde{\gamma}$ for the uniqueness of limit.

We have also that

$$\|\gamma_n\| = \frac{L_1^2}{\sigma\tau_n} + \frac{L_2^2}{(1-\sigma)(1-\tau_n)} \rightarrow \frac{L_1^2}{\sigma\tau} + \frac{L_2^2}{(1-\sigma)(1-\tau)} = \|\tilde{\gamma}\|, \quad (9)$$

so $\gamma_n \xrightarrow{H^1} \tilde{\gamma}$ and $a \leq \|\tilde{\gamma}\| \leq b$, hence $\tilde{\gamma} \in \Sigma_0$, that concludes the proof. \square

Now we shall prove two technical lemmas which are crucial for this paper.

Lemma 4 (existence of retraction in Σ_0). *There exist $R \supset \Sigma_0$, $\nu, \bar{t} \in \mathbb{R}^+$ and*

$$\eta_R : R \times [0, \bar{t}] \rightarrow \Omega$$

a continuous function s.t.

- $\eta_R(\beta, 0) = \beta$,
- $E(\eta_R(\beta, t)) - E(\beta) < -\nu t$,

for all $t \in [0, \bar{t}]$, $\beta \in R$.

Proof. We proceed by steps.

I) At first we want to prove that, for any $\gamma \in \Sigma_0$, there are $\bar{t}, d, \nu \in \mathbb{R}^+$, and a local retraction

$$\mathcal{H} : B(\gamma, d) \times [0, \bar{t}] \rightarrow \Omega$$

such that

- $\mathcal{H}(\beta, 0) = \beta$,
- $E(\mathcal{H}(\beta, t)) - E(\beta) < -\nu t$,

for all $t \in [0, \bar{t}]$, $\beta \in B(\gamma, d)$. Furthermore, we will see that d is independent from γ .

By hypothesis there exists a unique $\sigma \in [0, 1]$ such that $\gamma_0(\sigma) = v$; furthermore, because $E(\gamma_0) = c_0$, we know also that $|\gamma'_0|^2 = c_0$ almost everywhere. Let $\gamma \in \Sigma_0$, then $\text{Im}(\gamma) = \text{Im}(\gamma_0)$. In analogy with Lemma 3 we suppose, without loss of generality, that there exists a unique $\tau \in [0, 1]$ s.t. $\gamma(\tau) = v$, and both $\gamma'|_{(0, \tau)}$, $\gamma'|_{(\tau, 1)}$ are constant, although we cannot say if they are equals. We can choose a suitable change of parameter φ s.t.

$$\gamma(\varphi(s)) = \gamma_0(s). \quad (10)$$

By this way we can construct a flow for γ as follows:

$$\mathcal{H}(\gamma, t) = \gamma(\varphi_t(s)) = \begin{cases} \gamma\left(\frac{\tau}{a(t)}s\right) & s \in [0, a(t)) \\ \gamma\left(\frac{\tau-1}{a(t)-1}s + \frac{a(t)-\tau}{a(t)-1}\right) & s \in [a(t), 1] \end{cases} \quad (11)$$

where $a(t) = (1-t)\tau + t\sigma$.

Notice that

$$\gamma(\varphi_0(s)) = \gamma(s), \quad \gamma(\varphi_1(s)) = \gamma_0(s)$$

and

$$\gamma(\tau) = v = \gamma(\varphi_1(\sigma)) = \gamma(\varphi_t(a(t))).$$

We recall that $l(\gamma_0|_{(0,\sigma)}) = L_1$, $l(\gamma_0|_{(\sigma,1)}) = L_2$: obviously

$$\left[\frac{L_1}{\sigma}\right]^2 = \left[\frac{L_2}{1-\sigma}\right]^2 = c_0;$$

furthermore

$$\begin{aligned} \left[\frac{\partial}{\partial s}\gamma(\varphi_t(s))\right]_{(0,a(t))}^2 &= \left[\frac{L_1}{a(t)}\right]^2, \\ \left[\frac{\partial}{\partial s}\gamma(\varphi_t(s))\right]_{(a(t),1)}^2 &= \left[\frac{L_2}{1-a(t)}\right]^2, \end{aligned}$$

then

$$\begin{aligned} E(\mathcal{H}(\gamma, t)) &= \int_0^{a(t)} \frac{L_1^2}{a(t)^2} ds + \int_{a(t)}^1 \frac{L_2^2}{(1-a(t))^2} ds = \\ &= \frac{L_1^2}{\sigma^2} \frac{\sigma^2}{a(t)} + \frac{L_2^2}{(1-\sigma)^2} \frac{(1-\sigma)^2}{1-a(t)} = c_0 \left(\frac{\sigma^2}{a(t)} + \frac{(1-\sigma)^2}{1-a(t)} \right), \end{aligned}$$

so

$$\frac{\partial}{\partial t} E(\mathcal{H}(\gamma, t)) = c_0 \left(\frac{(1-\sigma)^2}{(1-a(t))^2} - \frac{\sigma^2}{(a(t))^2} \right). \quad (12)$$

It's easy to see that, either for $\sigma < \tau$ as for $\sigma > \tau$, we have $\frac{\partial}{\partial t} E(\mathcal{H}(\gamma, t)) < 0$, for all $t \in [0, 1)$, as expected. More over, because there is a $p > 0$ s.t. $p < \tau < 1-p$ (as shown in the previous lemma), we can find \bar{t}, ν s.t. $\frac{\partial}{\partial t} E(\mathcal{H}(\gamma, t)) < 2\nu \forall t \in [0, \bar{t}]$.

Now we want to extend \mathcal{H} in a neighborhood of γ : it's useful, for finding that, to work on the whole space $H^1(I, \mathbb{R}^n)$. As above we consider $\gamma \in \Omega$.

Let $B_d = B^{H^1(I, \mathbb{R}^n)}(\gamma, d) \cap \Omega$. For all $\beta \in B_d$ we can say

$$\beta = \gamma + (\beta - \gamma) = \gamma + \delta, \quad \|\delta\| \leq d$$

We can extend \mathcal{H} as follows:

$$\mathcal{H}(\beta, t) = \mathcal{H}(\gamma + \delta, t) = \gamma(\varphi_t(s)) + \delta(\varphi_t(s)) \quad (13)$$

Obviously $\text{Im}(\beta) = \text{Im}(\mathcal{H}(\beta, t))$, so $\mathcal{H}(\beta, t) \in \Omega$.

We want to show that there exists a $d > 0$ s.t.

$$E(\mathcal{H}(\beta, t)) - E(\beta) < -\nu t \quad \forall \beta \in B_d \quad (14)$$

$$\begin{aligned} E(\mathcal{H}(\beta, t)) - E(\beta) &= \\ &= \int |\gamma(\varphi_t(s))' + \delta(\varphi_t(s))'|^2 - \int |\gamma'(s) + \delta'(s)|^2 = \\ &= \int |\gamma(\varphi_t(s))'|^2 - |\gamma'(s)|^2 + \int |\delta(\varphi_t(s))'|^2 - |\delta'(s)|^2 + \\ &\quad + \int \langle \gamma(\varphi_t(s))', \delta(\varphi_t(s))' \rangle - \int \langle \gamma'(s), \delta'(s) \rangle. \end{aligned}$$

We have already shown that

$$\int |\gamma(\varphi_t(s))'|^2 - |\gamma'(s)|^2 < -2\nu t. \quad (15)$$

Let

$$A = \int_0^1 |\delta(\varphi_t(s))'|^2 ds - \int_0^1 |\delta(s)'|^2 ds$$

and

$$B = \int_0^1 \langle \gamma(\varphi_t(s))', \delta(\varphi_t(s))' \rangle ds - \int_0^1 \langle \gamma'(s), \delta'(s) \rangle ds.$$

The term A can be estimate as follows, remembering the definition of $\varphi_t(s)$:

$$\begin{aligned} A &= \\ &= \int_0^{a(t)} \left| \delta \left(\frac{\tau}{a(t)} s \right)' \right|^2 + \int_{a(t)}^1 \left| \delta \left(\frac{\tau-1}{a(t)-1} s + \frac{a(t)-\tau}{a(t)-1} \right)' \right|^2 - \int_0^1 |\delta'(s)|^2 = \\ &= \left[\frac{\tau}{a(t)} \right]^2 \int_0^{a(t)} \left| \delta' \left(\frac{\tau}{a(t)} s \right) \right|^2 + \\ &\quad + \left[\frac{\tau-1}{a(t)-1} \right]^2 \int_{a(t)}^1 \left| \delta' \left(\frac{\tau-1}{a(t)-1} s + \frac{a(t)-\tau}{a(t)-1} \right) \right|^2 - \int_0^1 |\delta'(s)|^2 = \\ &= \frac{\tau}{a(t)} \int_0^\tau |\delta'(s)|^2 + \frac{\tau-1}{a(t)-1} \int_\tau^1 |\delta'(s)|^2 - \int_0^1 |\delta'(s)|^2 = \\ &= \frac{\tau-a(t)}{a(t)} \int_0^\tau |\delta'(s)|^2 + \frac{\tau-a(t)}{a(t)-1} \int_\tau^1 |\delta'(s)|^2 \leq \\ &\leq \max \left[\left| \frac{\tau-a(t)}{a(t)} \right|, \left| \frac{\tau-a(t)}{a(t)-1} \right| \right] \int_0^1 |\delta'(s)|^2 \leq \\ &\leq |\tau-a(t)| \max \left[\frac{1}{a(t)}, \frac{1}{a(t)-1} \right] \int_0^1 |\delta'(s)|^2 \leq \\ &\leq K \|\delta\|_{H^1}^2 \leq K d^2 \cdot t \end{aligned}$$

in fact $|\tau-a(t)| = |\tau-\sigma|t$. Furthermore, K depends only on γ_0 , because $\exists p > 0$ s.t. $\tau \in [p, 1-p]$ (as shown in Lemma 3).

In the same way we can estimate B :

$$\begin{aligned} B &= \\ &= \frac{\tau-a(t)}{a(t)} \int_0^\tau \langle \gamma'(s), \delta'(s) \rangle + \frac{\tau-a(t)}{a(t)-1} \int_\tau^1 \langle \gamma'(s), \delta'(s) \rangle \leq \\ &\leq \left| \frac{\tau-a(t)}{a(t)} \right| \int_0^\tau |\langle \gamma'(s), \delta'(s) \rangle| + \left| \frac{\tau-a(t)}{a(t)-1} \right| \int_\tau^1 |\langle \gamma'(s), \delta'(s) \rangle| \leq \\ &\leq t|\tau-\sigma| \max \left[\frac{1}{a(t)}, \frac{1}{1-a(t)} \right] \int_0^1 \langle \gamma', \delta' \rangle \leq \\ &\leq K_1 d \cdot t, \end{aligned}$$

where, as above, K_1 is a constant depending only on γ_0 .

Now, putting together all the pieces we have

$$E(\mathcal{H}(\beta, t)) - E(\beta) \leq -2\nu t + Kd^2t + K_1dt < -\nu t \quad (16)$$

if $d < \min\left(\frac{\nu}{K_1}, \sqrt{\frac{\nu}{K}}\right)$.

II) We want to prove that, for all ε it exist a $0 < \tilde{t} < \bar{t}$ s.t.

$$\mathcal{H}(B(\beta, d'), t) \subset B(\beta, (1 + \varepsilon)d') \quad (17)$$

if $B(\beta, d') \subset B(\gamma, d)$, $t < \tilde{t}$. We start proving that, for any $\beta, \beta_1 \in B(\gamma, d)$,

$$\begin{aligned} \|\mathcal{H}(\beta, t) - \mathcal{H}(\beta_1, t)\|_{H^1}^2 &\leq \left(\frac{\tau}{a(t)}\right)^2 \int_0^{a(t)} |\beta' - \beta_1'|^2 \left(\frac{\tau}{a(t)}s\right) ds + \\ &\quad + \left(\frac{\tau-1}{a(t)-1}\right)^2 \int_{a(t)}^1 |\beta' - \beta_1'|^2 \left(\frac{\tau-1}{a-1}s + \frac{a-\tau}{a-1}\right) ds = \\ &= \left(\frac{\tau}{a(t)}\right) \int_0^\tau |\beta' - \beta_1'|^2(r) dr + \\ &\quad + \left(\frac{\tau-1}{a(t)-1}\right) \int_\tau^1 |\beta' - \beta_1'|^2(r) dr \leq \\ &\leq \max\left(\frac{\tau}{a(t)}, \frac{\tau-1}{a(t)-1}\right) \int_0^1 |\beta' - \beta_1'|^2(r) dr \leq \\ &\leq M^2(t) \|\beta - \beta_1\|_{H^1}^2 \quad \forall t. \end{aligned}$$

where $M(t)$ is a continuous function s.t. $M(0) = 1$.

In particular $\forall \varepsilon > 0$ there exists $\tilde{t} > 0$ s.t. for $t \leq \tilde{t}$

$$d(\mathcal{H}(\beta, t), \mathcal{H}(\beta_1, t)) < \left(1 + \frac{\varepsilon}{2}\right) \|\beta - \beta_1\|_{H^1}. \quad (18)$$

So, if $\beta_1 \in B(\beta, d')$, for all $\varepsilon > 0$ a \tilde{t} exists s.t.

$$\begin{aligned} d(\mathcal{H}(\beta_1, t), \beta) &\leq \\ &\leq d(\mathcal{H}(\beta_1, t), \mathcal{H}(\beta, t)) + d(\mathcal{H}(\beta, t), \beta) \leq \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) d' + \frac{\varepsilon}{2} d' \\ &\leq (1 + \varepsilon) d' \quad \forall 0 \leq t \leq \tilde{t}, \end{aligned}$$

because $\mathcal{H}(\beta, t)$ is continuous in t . Notice that \tilde{t} is independent from β_1 , so we have that, chosen β and d' s.t. $B(\beta, d') \subset B(\gamma, d)$, then for every $\varepsilon > 0$ there exists $\tilde{t} > 0$ such that

$$\mathcal{H}(B(\beta, d'), t) \subset B(\beta, (1 + \varepsilon)d') \quad \forall 0 \leq t \leq \tilde{t} \quad (19)$$

III) We have now to compound all these retraction. We follow an idea shown by Corvellec, Degiovanni and Marzocchi in [CDM93, theorem 2.8], and we combine it with the compactness of Σ_0 .

Take d as in the first step. Then $\bigcup_\gamma B(\gamma, d/4)$ covers Σ_0 . By compactness we can choose

$$\gamma_1, \dots, \gamma_N \text{ s.t. } \bigcup_{i=1}^N B\left(\gamma_i, \frac{d}{4}\right) \supset \Sigma_0.$$

Set $B(\gamma_i, d/4) = B_i$, $R = \bigcup_i \overline{B_i}$ and $\nu = \min_i \nu_{\gamma_i}$ and $\mathcal{H}_i = \mathcal{H}_{\gamma_i}$. Let

$$\vartheta_i : H^1(I, M) \rightarrow [0, 1]$$

a partition of unity referred to B_i .

We want to define a sequence of continuous maps

$$\eta_h : R \times [0, \tilde{t}_h] \rightarrow \Omega,$$

for $h = 1, \dots, N$, defined as follows:

$$\eta_1(\beta, t) = \begin{cases} \mathcal{H}_1(\beta, \vartheta_1 t), & \beta \in \overline{B_1} \\ \beta, & \text{outside;} \end{cases} \quad (20)$$

$$\eta_h(\beta, t) = \begin{cases} \mathcal{H}_h(\eta_{h-1}(\beta, t), \vartheta_h t), & \beta \in \overline{B_h} \\ \eta_{h-1}(\beta, t), & \text{outside.} \end{cases} \quad (21)$$

We want that, for all h ,

1. $\eta_h(\beta, 0) = \beta$;
2. $E(\eta_h(\beta, 0)) - E(\beta) \leq -\nu t \sum_{i=1}^h \vartheta_i$;
3. $\forall i, \forall \varepsilon \exists \tilde{t}_h$ s.t. $\eta_{h-1}(\overline{B_i}, t) \subset B(\gamma_i, (1 + \varepsilon)^{h-1} d/4)$ if $0 \leq t \leq \tilde{t}_h$.

The proof of the first two condition is obvious. The last condition, that assures the good definition of η_h , will be proved by induction on h .

a) Case $h = 1$:

If $B_i = B_1$ then $\eta_1(\beta, t) = \mathcal{H}_1(\beta, \vartheta_1 t)$. Hence there exists \tilde{t} s.t., if $0 \leq t \leq \tilde{t}$

$$d(\gamma_1, \eta_1(\beta, t)) = d(\gamma_1, \mathcal{H}_1(\beta, \vartheta_1 t)) < (1 + \varepsilon) \frac{d}{4}, \quad (22)$$

in fact we know that there exists \tilde{t} s.t.

$$d(\gamma_1, \mathcal{H}_1(\beta, t)) < (1 + \varepsilon) \frac{d}{4}, \quad (23)$$

for all $0 \leq t \leq \tilde{t}$, and $\vartheta_1 \leq 1$, so $\vartheta_1 t \leq t \leq \tilde{t}$.

If $B_1 \cap B_i = \emptyset$, then

$$\eta_1(\overline{B_i}, t) = \overline{B_i} \quad \forall t, \quad (24)$$

so the proof is obvious.

Finally, if $B_1 \cap B_i \neq \emptyset$, we know that

$$\overline{B} \left(\gamma_i, \frac{d}{4} \right) \subset B(\gamma_1, d),$$

so we can say that

$$\eta_1(\overline{B_i}, t) = \mathcal{H}_1(\overline{B_i}, \vartheta_1 t),$$

hence we can repeat the above deduction. Taking the minimum of \tilde{t} so found (they are a finite number) we can conclude.

b) Inductive step.

Let η_{h-1} be s.t., given $\varepsilon > 0$, for all i there exists a \tilde{t}_{h-1} for which

$$\eta_{h-1}(\overline{B}_i, t) \subset B\left(\gamma_i, (1 + \varepsilon)^{h-1} \frac{d}{4}\right) \quad \forall 0 \leq t \leq \tilde{t}_h. \quad (25)$$

At first notice that we can choose ε s.t.

$$\eta_{h-1}(\overline{B}_h, t) \subset B(\gamma_n, d),$$

so η_h is well defined.

Either if $B_i = B_h$ or if $B_h \cap B_i = \emptyset$ the proof is obvious.

Let $B_h \cap B_i \neq \emptyset$; if $\beta \in B_i \setminus \overline{B}_h$ then $\eta_h(\beta, t) = \eta_{h-1}(\beta, t)$, so

$$d(\gamma_h, \eta_h(\beta, t)) < (1 + \varepsilon)^{h-1} \frac{d}{4} < (1 + \varepsilon)^h \frac{d}{4}. \quad (26)$$

Otherwise, by inductive step

$$\eta_{h-1}(\overline{B}_h, t) \subset B\left(\gamma_h, (1 + \varepsilon)^{h-1} \frac{d}{4}\right),$$

and, by (19) we have that there exists a \tilde{t} s.t

$$\mathcal{H}_h(\eta_{h-1}(\beta, t)) \subset B\left(\gamma_h, (1 + \varepsilon)^h \frac{\delta}{4}\right) \quad (27)$$

if $\beta \in \overline{B}_h \cap B_i$ and $0 \leq t \leq \tilde{t}$, so the proof follows immediately. Because we have N iterations, we choose $\bar{\varepsilon}$ s.t. $(1 + \bar{\varepsilon})^N < 2$, and we define

$$\bar{t} = \min_h \{t_{\bar{\varepsilon}, h} \text{ previously found } \}. \quad (28)$$

By compactness $\bar{t} > 0$. Set

$$\eta_R = \eta_N, \quad (29)$$

so we find a continuous map

$$\eta_R : R \times [0, \bar{t}] \rightarrow \Omega \quad (30)$$

such that

- $\eta_R(\beta, 0) = \beta$,
- $E(\eta_t(\beta, t)) - E(\beta) \leq -\nu t \sum_{i=1}^N \vartheta_i = -\nu t$,

for every $\beta \in R$, $0 \leq t \leq \bar{t}$. □

Lemma 5. For any $U \supset \Sigma_0$ there exist $\bar{t}, \nu \in \mathbb{R}^+$ and a continuous functional

$$\eta_U : \Omega_a^b \setminus U \times [0, \bar{t}] \rightarrow \Omega_a^b$$

such that

- $\eta_U(\cdot, 0) = Id$,
- $E(\eta_U(\beta, t)) - E(\beta) \leq -\nu t$,

for all $t \in [0, \bar{t}]$, for all $\beta \in \Omega_a^b \setminus U$

Proof. We look for a pseudo gradient vector field F , s.t., if η_U is a solution of

$$\begin{cases} \dot{\eta}_U(t, \gamma) = F(\gamma) \\ \eta_U(0, \cdot) = \text{Id} \end{cases} \quad (31)$$

then

$$\exists \nu > 0 \text{ s.t. } E(\eta_U(t, \gamma)) - E(\gamma) < -\nu t. \quad (32)$$

For every S neighborhood of Σ , we have that $-\nabla E$ is a good gradient field on $\Omega_a^b \setminus S$, in fact E is smooth and satisfies the Palais Smale condition outside S , so, for $\Omega_a^b \setminus S$ does not contain critical points of E , we know that there exists a $\nu_0 \in \mathbb{R}^+$ s.t.

$$-||\nabla E||^2 < -\nu_0; \quad (33)$$

by integrating (31) with $F = -\nabla E$ we have that

$$E(\eta_U(\gamma, t)) - E(\gamma) < -\nu_0 t \quad (34)$$

if $\eta_U(\gamma, t) \subset \Omega_a^b \setminus S$ for all t .

Now let S_1 be a neighborhood of S and let U be a neighborhood of Σ_0 : we look for a pseudo gradient vector field on $S_1 \setminus U$. Although E is non smooth, we can define $dE(\gamma)[w]$ for every $\gamma \in \Sigma \setminus U$, and for a suitable choice of w . It is sufficient to take w vector field along γ with

$$\text{spt } w \subset \{s \text{ s.t. } \gamma(s) \neq v\}.$$

There exists ν_1 such that for every $\gamma \in \Sigma \setminus U$ we can find w_γ for which $dE(\gamma)[w_\gamma] < -2\nu_1$. This is possible because we can find a partition $0 = s_0 < \dots < s_k = 1$ such that $\gamma(s_i) = v$ and $v \notin \text{Im } \gamma|_{(s_i, s_{i+1})}$. Called $\gamma_i = \gamma|_{(s_i, s_{i+1})}$ we can shorten it by a vector field w_i along γ_i , leaving its extremal point fixed, so we obtain a vector field w_γ along γ with

$$\text{spt } w_\gamma \subset \{s \text{ s.t. } \gamma(s) \neq v\},$$

and

$$dE(\gamma)[w_\gamma] < -2\nu_1, \quad (35)$$

in fact for these variations the (P.S.) condition for energy holds. Moreover, $\Sigma \setminus U$ does not contain any stationary point for these kind of variations.

Without loss of generality suppose now that exists a global chart (V, ϕ) , $0 \in V \subset \mathbb{R}^n$ s.t. $\phi(0) = v$. The metric of M , read on V , lead us to consider a matrix $(g_{ij}(x))_{ij}$ whose coefficients are discontinuous at 0; if γ is a path on V we can compute its energy by taking

$$E(\gamma) = \int g_{ij}(\gamma) \gamma'_i \gamma'_j ds. \quad (36)$$

For the sake of simplicity we suppose also that

$$g_{ij}(x) = g(x) \delta_{ij}(x)$$

where δ_{ij} are the coefficient of Euclidean metric. The general case does not present further difficulties.

Now we pass to coordinates (V, ϕ) . Because $\gamma \in \Omega$, if $\|\gamma - \gamma_1\|_\Omega < \varepsilon$ then there exists $C \in \mathbb{R}^+$ s.t. $\|\gamma - \gamma_1\|_{L^\infty} < C\varepsilon$ by the Sobolev immersion, so also $\|\phi(\gamma) - \phi(\gamma_1)\|_{L^\infty} < C\varepsilon$.

In coordinates $dE(\gamma)[w]$ has the following form:

$$dE(\gamma)[w] = \int g(\gamma) \gamma' w' ds + \int \langle \nabla g, w \rangle |\gamma'|^2 ds, \quad (37)$$

where $w \in H^1(I, V)$. Note that, even if ∇g does not exist everywhere, it is well defined on $\text{spt } w$.

We have proved that for every $\gamma \in \Sigma \setminus U$ exists w_γ s.t. $dE(\gamma)[w_\gamma] < -2\nu_1$; obviously we can prove the same for every $\gamma \in S_1 \setminus U$. Given $\gamma \in S_1 \setminus U$ and w_γ as above, it exists a neighborhood V_γ of γ s.t.

$$\forall \gamma_1 \in V_\gamma \quad dE(\gamma_1)[w_\gamma] < -\nu_1. \quad (38)$$

Let $\|\gamma - \gamma_1\|_{H^1} < \varepsilon$, then

$$\begin{aligned} & \int g(\gamma) \gamma' w'_\gamma - \int g(\gamma_1) \gamma'_1 w'_\gamma \leq \\ & \leq \int g(\gamma) (\gamma' - \gamma'_1) w'_\gamma + \int (g(\gamma) - g(\gamma_1)) \gamma'_1 w'_\gamma \leq \\ & \leq \sup_{t \in \text{spt } w_\gamma} g(\gamma) \|\gamma' - \gamma'_1\|_{L^2} \|w'\|_{L^2} + \sup_{t \in \text{spt } w_\gamma} [g(\gamma) - g(\gamma_1)] \|\gamma'_1\|_{L^2} \|w'\|_{L^2} \leq \\ & \leq \text{Const} \cdot \varepsilon, \end{aligned}$$

in fact $g(\gamma) \in C^\infty(\text{spt } w_\gamma)$, so $\sup g(\gamma)$ is bounded; furthermore, because $\|\gamma - \gamma_1\|_{L^\infty} < C \cdot \varepsilon$, $\sup [g(\gamma) - g(\gamma_1)] \leq C \cdot \varepsilon$.

In the same way

$$\begin{aligned} & \int \langle \nabla g(\gamma), w_\gamma \rangle |\gamma'|^2 - \int \langle \nabla g(\gamma_1), w_\gamma \rangle |\gamma'_1|^2 \leq \\ & \int \langle \nabla g(\gamma), w_\gamma \rangle (|\gamma'|^2 - |\gamma'_1|^2) + \int \langle \nabla g(\gamma) - \nabla g(\gamma_1), w_\gamma \rangle |\gamma'_1|^2 \leq \\ & \leq \text{Const} \cdot \varepsilon. \end{aligned}$$

So $dE(\gamma)[w_\gamma] - dE(\gamma_1)[w_\gamma] \leq C \cdot \varepsilon$: we can choose a neighborhood V_γ , for all $\gamma \in S_1 \setminus U$, s.t.

$$dE(\gamma_1)[w_\gamma] < -\nu_1 \quad \forall \gamma_1 \in V_\gamma. \quad (39)$$

The sets V_γ covers the whole $S_1 \setminus U$. Let V_{γ_i} be a locally finite refinement of V_γ . Let β_i be a partition of the unity associated to V_{γ_i} . Then

$$F_1 = \sum \beta_i w_{\gamma_i} \quad (40)$$

is a pseudo-gradient vector field on $S_1 \setminus U$ (for the details of such a construction see [Rab74]). Now let α_j be a partition of the unity associated to $S_1 \setminus U, \Omega_a^b \setminus S$, then

$$F = \alpha_1 F_1 - \alpha_2 \nabla E \quad (41)$$

is the vector field we looked for, in fact we can find η_U because F is a Lipschitz vector field by definition. Even if E isn't smooth, we can differentiate it along

the direction of F , so

$$\begin{aligned} E(\eta_U(\gamma, t)) - E(\gamma) &= \int_0^t \frac{d}{d\tau} E(\eta_U(\gamma, \tau)) d\tau = \\ &= \int_0^t dE(\eta_U(\gamma, \tau)) [\dot{\eta}_U(\gamma, \tau)] d\tau = \int_0^t dE(\eta_U(\gamma, \tau)) [F]. \end{aligned}$$

Let $\nu = \min(\nu_0, \nu_1)$. Then

$$\begin{aligned} E(\eta_U(\gamma, t)) - E(\gamma) &= \\ &= \int_0^t \alpha_1 dE(\eta_U(\gamma, \tau)) [F_1] - \alpha_2 \|\nabla E(\eta_U(\gamma, \tau))\|^2 = \\ &= \int_0^t \alpha_1 \sum \beta_i dE(\eta_U(\gamma, \tau)) [w_{\gamma_i}] - \alpha_2 \|\nabla E(\eta_U(\gamma, \tau))\|^2 \leq \\ &\leq \int_0^t -\alpha_1 \sum \beta_i \nu_1 - \alpha_2 \nu_0 \\ &\leq -\int_0^t \nu \leq -\nu t. \end{aligned}$$

□

From lemma 4 and lemma 5 we get the following result.

Theorem 6. *Let M be a conical manifold with only a vertex v , and consider the special closed geodesic γ_0 for which there exists a unique σ s.t. $\gamma_0(\sigma) = v$. Set $E(\gamma_0) = c_0$. Suppose that there exist $a, b \in \mathbb{R}$, $c_0 < a < b$ s.t. Ω^b contains only the geodesics γ_0 .*

Then $\Omega^b \simeq \Omega^a$.

Proof. Given R as in lemma 4, we choose U and V neighborhoods of Σ_0 s.t.

$$\Sigma_0 \subsetneq U \subsetneq V \subsetneq R.$$

we know that, for such an U , there exists a retraction η_U defined as in Lemma 5. For the sake of simplicity we will suppose that η_U and η_R (see Lemma 4) are defined for $0 \leq t \leq 1$ and that ν is the same for both of them. Let $\theta_1 : \Omega^b \rightarrow [0, 1]$ a continuous map s.t.

$$\begin{aligned} \theta_1|_U &\equiv 0 \\ \theta_1|_{\Omega^b \setminus V} &\equiv 1. \end{aligned}$$

Then we define a continuous map

$$\mu_1 : \Omega^b \times [0, 1] \rightarrow \Omega^b, \quad (42)$$

$$\mu_1(\beta, t) = \eta_U(\beta, \theta_1(\beta)t); \quad (43)$$

we know that $E(\mu_1(\beta, t)) - E(\beta) \leq -\nu t \theta_1(\beta)$, so

$$\mu_1(\Omega^b, 1) \subset V \cup \Omega^{b-\nu},$$

in fact if $\mu_1(\beta, t) \notin V$ for all t , then $E(\mu_1(\beta, t)) - E(\beta) \leq -\nu t$, so $\mu_1(\beta, 1) \in \Omega^{b-\nu}$.

By μ_1 we have retracted Ω^b on $\Omega^{b-\nu} \cup V$; now we define a continuous map $\theta_2 : \Omega^b \rightarrow [0, 1]$ s.t.

$$\begin{aligned}\theta_1|_{\Omega_{b-\nu/2}^b} &\equiv 1, \\ \theta_1|_{\Omega^{b-\nu}} &\equiv 0.\end{aligned}$$

Then set

$$\mu_2 : V \cup \Omega^{b-\nu} \times [0, 1] \rightarrow \Omega^b \quad (44)$$

$$\mu_2(\beta, t) = \eta_R(\beta, \theta_2(\beta)t); \quad (45)$$

μ_2 is a continuous map that retracts $V \cup \Omega^{b-\nu}$ on $\Omega^{b-\nu/2}$. By iterating this algorithm we can retract continuously Ω^b on Ω^a . \square

Now we can prove the deformation lemma.

Proof of Theorem 2. Let $\{\gamma_i\}_{i=1, \dots, N}$ be the set of geodesics in Ω^b . We start defining some special subset of Ω_a^b , as in (3) and (4); let

$$\Sigma = \{\gamma \in \Omega_a^b, \text{ s.t. } \text{Im} \gamma \cap V \neq \emptyset\} \quad (46)$$

(we recall that V is the set of vertexes); for $i = 1, \dots, N$, set

$$\Sigma_i = \{\gamma \in \Sigma \text{ s.t. } \gamma = \gamma_i \text{ up to affine reparametrization}\}. \quad (47)$$

We note that for $i \neq j$ then $\Sigma_i \cap \Sigma_j = \emptyset$, because the geodesics are different. For these Σ_i we can find a retraction η_Σ as in lemma 4: indeed, for every $U \supset \bigcup_{i=0}^N \Sigma_i$ there exists a retraction η_U on $\Omega_a^b \setminus U$ in analogy with lemma 5. Finally, we compound these two maps η_Σ and η_U following the proof of theorem 6 and we conclude. \square

Theorem 7 (Second deformation lemma). *Let M be a conical manifold, $p \in M$. Suppose that there exists $c \in \mathbb{R}$ s.t. Ω^c contains only a finite number of geodesics and that there exists $a, b \in \mathbb{R}$, $a < b < c$ s.t. the strip $[a, b)$ contains only regular values of E . Set Z the set of geodesics and $Z_b = Z \cap E^{-1}(b)$, then there exists a neighborhood U of Z_b s.t.*

$$\Omega^b \setminus U \simeq \Omega^a.$$

Proof. We can prove this corollary following the lines of Theorem 2. \square

As previously said, Lemma 4, which is crucial for this work, is based on a generalization of [CDM93, theorem 2.8]. Indeed, using a slight modification of the weak slope tool, this result and the deformation lemmas can be reformulated in a more general context. This theoretic frame is briefly discussed in the appendix.

3 Category theory

First, we recall some well known results relative to the Lusternik and Schnirelmann category. This theory was presented in [LS34] in a finite dimensional framework, then generalized to Banach manifold by R. Palais [Pal66b].

Definition 4. Let X be a topological space, $A \subset X$. If $A \neq \emptyset$ we say that

$$\text{cat } A = \text{cat}_X A = k \quad \text{iff}$$

k is the least integer for which there are F_1, \dots, F_k closed contractible subsets of X s.t. $\bigcup_k F_k$ covers A .

We define also

$$\text{cat } \emptyset = \text{cat}_X \emptyset = 0.$$

Theorem 8. Let X be a topological space. Then

1. if $A \subset B \subset X$ then $\text{cat}_X A \leq \text{cat}_X B$;
2. if $A, B \subset X$ then $\text{cat}_X A \cup B \leq \text{cat}_X A + \text{cat}_X B$;
3. if $A, B \subset X$, A closed, and there is $\eta \in C([0, 1] \times A, X)$ s.t.

$$\begin{aligned} B &= \eta(1, A); \\ \eta(0, u) &= u \quad \forall u \in A, \end{aligned}$$

then $\text{cat}_X A \leq \text{cat}_X B$

4. if Y is a topological space, $y \in Y$, then $\text{cat}_{X+Y}(A \times \{y\}) + \text{cat}_X A$.

Proof. The points 1,2 and 4 are trivial. We have only to prove 3.

By hypothesis, we can find F_1, \dots, F_k s.t. $B \subset F_1 \cup \dots \cup F_k$. Set

$$C_i = \{u \in A \text{ s.t. } \eta(1, u) \in F_i\}.$$

Obviously, C_i are closed and contractible. Since $C_1 \cup \dots \cup C_k$ covers A we obtain the thesis. \square

By Theorem 7 we are able to reconstruct the category theory for the energy functional defined on a conical manifold.

Lemma 9. Let M be a conical manifold, $p \in M$. Suppose that there exists $\bar{c} \in \mathbb{R}$ s.t. $\Omega^{\bar{c}}$ contains only a finite number of geodesics. Let $c < \bar{c}$ a critical level for E . Then, set U a neighborhood of Z_c there exists $\varepsilon > 0$ s.t.

$$\text{cat } \Omega^{c+\varepsilon} \leq \text{cat } \Omega^{c-\varepsilon} + \text{cat } U. \quad (48)$$

Proof. We know, by the second deformation lemma, that $\Omega^{c-\varepsilon}$ is a deformation retract of $\Omega^{c+\varepsilon} \setminus U$: applying Theorem 8 we obtain

$$\text{cat } \Omega^{c+\varepsilon} \leq \text{cat } \Omega^{c+\varepsilon} \setminus U + \text{cat } U \leq \text{cat } \Omega^{c-\varepsilon} + \text{cat } U. \quad (49)$$

\square

Theorem 10. Let M be a conical manifold, let $p \in M$ and let $a < b \in \mathbb{R}$. Then Ω_a^b contains at least $\text{cat } \Omega^b - \text{cat } \Omega^a$ geodesics.

Proof. We suppose that there is a finite number of critical levels in $[a, b]$ (otherwise there is nothing to prove). Set $a \leq c_0 < c_1 < \dots < c_k \leq b$ these critical levels, and set, for all i , U_i a neighborhood of Z_{c_i} . We know that there exists an ε s.t. for all i

$$\text{cat } \Omega^{c_i+\varepsilon} \leq \text{cat } \Omega^{c_i-\varepsilon} + \text{cat } U_i. \quad (50)$$

By iterating (50), and using the deformation lemma, we obtain

$$\begin{aligned} \text{cat } \Omega^{c_k+\varepsilon} &\leq \text{cat } \Omega^{c_k-\varepsilon} + \text{cat } U_k \leq \text{cat } \Omega^{c_{k-1}+\varepsilon} + \text{cat } U_k \leq \\ &\leq \text{cat } \Omega^{c_{k-2}+\varepsilon} + \text{cat } U_{k-1} + \text{cat } U_k \leq \dots \leq \\ &\leq \text{cat } \Omega^{c_0-\varepsilon} + \sum_{i=0}^k \text{cat } U_i. \end{aligned}$$

Because $\text{cat } \Omega^b \leq \text{cat } \Omega^{c_k+\varepsilon}$ and $\text{cat } \Omega^{c_0-\varepsilon} \leq \text{cat } \Omega^a$ we have that

$$\text{cat } \Omega^b - \text{cat } \Omega^a \leq \sum_{i=0}^k \text{cat } U_i. \quad (51)$$

Suppose now that there are a finite number of geodesics for any critical level. Because every point has a contractible neighborhood, we can choose U_i s.t.

$$\text{cat } U_i \leq \#Z_{c_i}, \quad (52)$$

thus

$$\text{cat } \Omega^b - \text{cat } \Omega^a \leq \sum_i \#Z_{c_i}. \quad (53)$$

□

From theorem 10 the main result of this paper follows.

Corollary 10.1. *Let M be a conical manifold, $p \in M$. Then there are at least $\text{cat } \Omega$ geodesics.*

Proof. If there is an infinite number of geodesics, there is nothing to prove.

Otherwise, we can apply the previous theorem and we conclude by a limiting process. (consider that $\Omega^{-1} = \emptyset$ and that $\Omega^b \simeq \Omega$ for $b \gg 1$). □

4 An application

We show a topological lemma necessary to provide some applications.

Let X a smooth submanifold of \mathbb{R}^n . Given $g \in L^\infty(X, \mathbb{R}^+)$, set

$$E(\gamma) = \int_0^1 g(\gamma(s)) |\gamma'|^2 ds.$$

We set

$$\begin{aligned} G(I, X) &= \{\gamma \in C^0(I, X) : E(\gamma) \text{ is well defined and finite}\}; \\ G(S^1, X) &= \{\gamma \in C^0(S^1, X) : E(\gamma) \text{ is well defined and finite}\}. \end{aligned}$$

Obviously we have that

$$H^1(I, X) \subset G(I, X) \subset C^0(I, X); \quad (54)$$

$$H^1(S^1, X) \subset G(S^1, X) \subset C^0(S^1, X). \quad (55)$$

We recall that

$$\begin{aligned} \Omega &= \Omega_p X = \{\gamma \in H^1([0, 1], X) : \gamma(0) = \gamma(1) = p\}; \\ \Omega^\infty &= \Omega_p^\infty X = \{\gamma \in C^0(S^1, X) : \gamma(0) = \gamma(1) = p\}, \end{aligned}$$

as previously defined. We define also the free loop space on X as

$$\begin{aligned} \Lambda &= \Lambda X = \{\gamma \in H^1(S^1, X)\}; \\ \Lambda^\infty &= \Lambda^\infty X = \{\gamma \in C^0(S^1, X)\}. \end{aligned}$$

In analogous way we set G (resp. G_p) the subspace of Λ^∞ (resp. Ω^∞) in which E is well defined and finite, according with previous definitions. These definitions allow us to formulate the following lemma.

Lemma 11. *Let X, g and $E(\cdot)$ be as above. Then*

$$\text{cat}_G G \geq \text{cat}_{\Lambda^\infty} \Lambda^\infty; \quad (56)$$

$$\text{cat}_{G_p} G_p \geq \text{cat}_{\Omega^\infty} \Omega^\infty. \quad (57)$$

In particular, if X is a connected and non contractible manifold then

$$\text{cat } G_p = \text{cat } \Omega^\infty = \infty. \quad (58)$$

Proof. We show only (56), then (57) follows in the same way. Because X is a smooth manifold, it is well known that there is an homotopic equivalence between Λ^∞ and Λ (see, e.g. [Kli78, Th 1.2.10]). Then

$$\text{cat}_{\Lambda^\infty} \Lambda = \text{cat}_{\Lambda^\infty} \Lambda^\infty;$$

now, because $\Lambda \subset G \subset \Lambda^\infty$, we have

$$\text{cat}_G G \geq \text{cat}_{\Lambda^\infty} \Lambda = \text{cat}_{\Lambda^\infty} \Lambda^\infty, \quad (59)$$

that proves (57)

Formula (58) is a standard result and can be found, for example in [FH91, Corollary 1.2] \square

By this result we can compute $\text{cat } \Omega$ in some concrete case, as shown in the next example.

Example 2. Let $M \subset \mathbb{R}^n$ a compact conical manifold, V the set of its vertexes. Suppose that there exists a compact smooth manifold $X \subset \mathbb{R}^k$ and an homeomorphism $\psi : M \rightarrow X$ s.t.

$$\psi|_{M \setminus V} \in C^\infty(M \setminus V, X),$$

then there exists g^* an induced metric on X defined by

$$g_p^*(v, w) := \begin{cases} g_{\psi^{-1}(p)}(d\psi^{-1}(v), d\psi^{-1}(w)) & \text{on } X \setminus \psi(V); \\ 0 & \text{otherwise.} \end{cases} \quad (60)$$

If $|d\psi^{-1}| \in L^\infty(X)$, we have that g^* is bounded with respect to the Euclidean metric of X and that

$$\Omega_{\psi^{-1}(p)}(M) = G_p(X) := \left\{ \gamma \in C^0([0, 1], X), \int_0^1 g_\gamma^* |\gamma'|^2 < \infty \right\}. \quad (61)$$

In this case, we can apply lemma 11 to compute the category of based (or free) loop space of M .

We also state an immersion theorem that is, in some sense, the converse of previous example.

Theorem 12 (Nash immersion for conical manifolds). *Let X a smooth manifold and let g a continuous non negative and bounded bilinear tensor s.t. there exist V a finite set of points and g is smooth and positive defined on $X \setminus V$. Then*

1. *If $V = \{x\}$, then, for N sufficiently large, there exists $M \subset \mathbb{R}^N$ a conical manifold and a continuous map*

$$\psi : X \rightarrow M$$

s.t $\psi|_{X \setminus V}$ is a C^∞ isometry.

2. *If $V = \{x_1, \dots, x_k\}$, for every x_i it exists $\rho_i > 0$ s.t. $B(x_i, \rho_i)$ is isometric (in the sense above specified) to some conical manifold $M_i \subset \mathbb{R}^N$.*

Proof. We start proving 1. By hypothesis, $(X \setminus V, g)$ is a Riemannian manifold, so, by Nash theorem [Nas56], it can be embedded in \mathbb{R}^N , for N sufficiently large. Let $\psi : X \setminus V \rightarrow M$ be this embedding.

We can continuously extend ψ to the whole X . In fact, let $\{x_n\}_n$ be a Cauchy sequence converging to x ; because g is bounded, then $\{\psi(x_n)\}_n$ is a Cauchy sequence in \mathbb{R}^N , so there exists $y \in \mathbb{R}^N$ s.t. $\lim \psi(x_n) = y$. Set $\psi(x) := y$: obviously we have that

$$\psi(B_X(x, \rho)) \subset B_{\mathbb{R}^N}(y, r), \quad (62)$$

and $r \xrightarrow{\rho \rightarrow 0} 0$, so ψ is continuous at x .

Then, set $M := \psi(X)$, we have that M is a conical manifold with vertex y , isometric to X .

To proof 2, it is sufficient to choose ρ_i s.t. $B(x_i, \rho_i)$ are all disjoint. Then we apply the previous result with $X = B(x_i, \rho_i)$. \square

By this result, we formulate a result which will be useful in the next of this paper.

Theorem 13. *In the above hypothesis, we have that*

$$\text{number of geodesics in } X \geq \text{cat } G$$

Proof. If X has an unique vertex, it is isometric to a conical manifold M . Then, by applying lemma 11, we obtain the proof. If the manifold X has several vertexes, we are in the case 2 of previous theorem.

Anyway, by the local isometries, we can prove an analogous of deformation lemma for geodesics in X . Also an analogous of theorem 10 follows. This, paired with lemma 11 gives us the proof. \square

4.1 Brachistochrones

In this section we want to study the brachistochrones problem. A brachistochrone is a curve γ which minimizes the time of transit for a particle moving from a point p towards a point q . We study this problem on $(S^n, <, >)$ an Euclidean sphere embedded in \mathbb{R}^{n+1} . We suppose that the particle moves in the presence of a potential $U : S^n \rightarrow \mathbb{R}$ without friction. Also, we are interested to any curve stationary for the time of transit functional.

Be $p, q \in S^n$, $E \in \mathbb{R}^+$ the energy of the particle, $U \in C^\infty(S^n, \mathbb{R})$ the given potential. It is well known that, if there exist $c_1, c_2 \in \mathbb{R}$ s.t.

$$-\infty < c_1 < U(\cdot) < c_2 < E, \quad (63)$$

then this problem is equivalent to the geodesic problem for the Riemannian manifold

$$\left(S^n, g := g_x = \frac{<, >}{E - U(x)} \right),$$

and that the metric g is equivalent to the Euclidean metric on S^n , so the problem has always a solution. Furthermore, it is also well known that a solution exists even if the upper bound on $U(x)$ does not exist.

In this section we want to study the problem for a given potential

$$U \in C^\infty(S^n \setminus V, \mathbb{R})$$

where $V = \{x_1, \dots, x_k\}$ a finite set of points on the sphere, and

$$U(x) \xrightarrow{x \rightarrow x_i} -\infty$$

As we will see in the next section, potential in S^n with these kind of singularities may appear from non singular potential defined in \mathbb{R}^n .

For the sake of simplicity, we suppose that there exist $c > 0$ for which $E > c > U(\cdot)$.

We define a metric on S^n by

$$g := g_x = \begin{cases} \frac{<, >}{E - U(x)} & \text{on } S^n \setminus V; \\ 0 & \text{otherwise,} \end{cases} \quad (64)$$

and we look for g_x -geodesics between two given points $p, q \in S^n$. Set, as usual

$$\begin{aligned} G(p) &= \left\{ \gamma \in C^0([0, 1], S^n), \gamma(0) = \gamma(1) = p, \frac{1}{2} \int_0^1 g(\gamma', \gamma') < \infty \right\}; \\ G(p, q) &= \left\{ \gamma \in C^0([0, 1], S^n), \gamma(0) = p, \gamma(1) = q, \frac{1}{2} \int_0^1 g(\gamma', \gamma') < \infty \right\}, \end{aligned}$$

we know that g satisfies the hypothesis of lemma 11, so

$$\text{cat } G(p) = \infty. \quad (65)$$

It's easy to prove that there is an homotopy equivalence between $G(p)$ and $G(p, q)$, in fact, for any given couple of points p, q , there exists a continuous

curve γ which joins them, with $E(\gamma) < \infty$ (because g is bounded). Then there is a map

$$\begin{aligned} i &: G(p) \rightarrow G(p, q); \\ \beta &\mapsto \beta + \gamma, \end{aligned}$$

where $\beta + \gamma$ is the usual composition of paths.

Of course there exists the inverse map

$$\begin{aligned} i^{-1} &: G(p, q) \rightarrow G(p); \\ \beta &\mapsto \beta + (-\gamma). \end{aligned}$$

and $i^{-1} \circ i$ is homotopic equivalent to $1_{G(p)}$.

By the above consideration and by Nash theorem we have that

$$\infty = \text{cat } G(p) = \text{cat } G(p, q) = \text{number of geodesics between } p \text{ and } q,$$

thus we can count the number of brachistochrones on the sphere in presence of our potential U .

4.1.1 Brachistochrones in \mathbb{R}^n

A more interesting application is the study of the same brachistochrone problem in \mathbb{R}^n (indeed this was the very beginning of our research). Let $U \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and $E > 0$ s.t.

- $E > U(x)$,
- $-U(x) = O(|x|^\alpha)$ when $|x| \gg 1$, for some $\alpha > 0$.

We are looking for brachistochrones joining two given points $p, q \in \mathbb{R}^n$ in presence of potential $U(x)$. As above we look for geodesics in

$$\left(\mathbb{R}^n, g_x := \frac{<, >}{E - U(x)} \right), \quad (66)$$

where $\frac{1}{E - U(x)} \in C^\infty(\mathbb{R}^n, \mathbb{R} \setminus \{0\}) \cap L^\infty(\mathbb{R}^n)$.

We can map \mathbb{R}^n in $S^n \subset \mathbb{R}^{n+1}$ by the stereographic map π . The inverse map is

$$\pi^{-1} : \begin{cases} S^n \setminus N \subset \mathbb{R}^{n+1} & \rightarrow & \mathbb{R}^n \\ \begin{pmatrix} y_1 \\ \dots \\ y_{n+1} \end{pmatrix} & \mapsto & \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{y_1}{1 - y_{n+1}} \\ \dots \\ \frac{y_n}{1 - y_{n+1}} \end{pmatrix} \end{cases}, \quad (67)$$

where N is the north pole of S^n . As usual we can induce a metric g^* on S^n defined by

$$g^*(y)(v, w) = \begin{cases} g_{\pi^{-1}(y)}(d\pi^{-1}v, d\pi^{-1}w) & \text{on } S^n \setminus N, \\ 0 & y = N. \end{cases} \quad (68)$$

It's easy to see that

$$|d\pi^{-1}| = O\left(\frac{1}{\sqrt{1 - y_{n+1}}}\right), \quad (69)$$

that, read on \mathbb{R}^n , becomes

$$|d\pi^{-1}| = O\left(\frac{1}{|x|}\right). \quad (70)$$

So, if $\alpha > 2$, then g^* is bounded with respect to the Euclidean metric on S^n , and we can apply lemma 11.

Furthermore, by Nash embedding, there is an isometry with a compact conical manifold, so we can easily state that there is an infinite number of brachistochrones joining p and q , although we cannot say if they are bounded in \mathbb{R}^n , and so physical meaningful.

A The theoretic frame

As said, our deformation lemmas (Lemma 2 and Lemma 7) are obtained modifying a weak slope theory result. In this section we present the k -slope, a generalization of the weak slope, which allows us to reformulate the main results of this paper in a more general framework.

We start recalling the definition of weak slope.

Definition 5. Let (X, d) be a metric space and let $f : X \rightarrow \mathbb{R}$ be a continuous functional. The weak slope of f at $u \in X$ (noted $|df|(u)$) is the supremum of σ 's in $[0, +\infty)$ s.t. $\exists \delta > 0$ and $\mathcal{H} : B(u, \delta) \times [0, \delta] \rightarrow X$ continuous with

$$d(\mathcal{H}(v, t), v) \leq t \quad (71)$$

$$f(\mathcal{H}(v, t)) - f(v) \leq -\sigma t \quad (72)$$

for every $v \in B(u, \delta)$, $t \in [0, \delta]$.

Due to (71) we can prove a deformation property for continuous functionals ([CDM93, theorem 2.8]): this inequality allows us to compound the local maps \mathcal{H} finding a global retraction.

Unhappily, these tools are not completely useful for our purposes. In particular we was not able to prove an estimate like (71). In our work we override these difficulties using the compactness of sets Σ_i and compounding explicitly all the local retractions. This method has a generalization that we present here.

A.1 The k -slope

We define an extension of weak slope which will be called k -slope.

Definition 6. Let (X, d) be a metric space. Let $f : X \rightarrow \mathbb{R}$ be a continuous functional and let $u \in X$. We define the k -slope of f at $u \in X$ (noted $|d_k f|(u)$) as the supremum of $\sigma \in [0, \infty)$ s.t. exist $\delta > 0$, $k_u : [0, \delta] \rightarrow \mathbb{R}^+$ continuous, $k_u(0) = 0$, and a continuous map

$$\mathcal{H} : B(u, \delta) \times [0, \delta] \rightarrow X$$

which satisfies

$$d(\mathcal{H}(v, t), v) \leq k_u(t) \quad (73)$$

$$f(\mathcal{H}(v, t)) - f(v) \leq -\sigma t \quad (74)$$

for all $v \in B(u, \delta)$, for all $t \in [0, \delta]$

In analogy with the weak slope theory we can prove the following property.

Proposition 14. *If f is continuous, $|d_k f|$ is lower semi-continuous.*

Proof. If $|d_k f|(u) = 0$ the proof is obvious. Otherwise, for any $0 < \sigma < |d_k f|(u)$ there exist δ and $\mathcal{H} : B(u, \delta) \times [0, \delta] \rightarrow X$ as in definition 6. Let $u_h \rightarrow u$. Definitively we have $u_h \in B(u, \frac{\delta}{2})$, so we can take the restriction of \mathcal{H} to $B(u_h, \frac{\delta}{2}) \times [0, \frac{\delta}{2}]$, to have $|d_k f(u_h)| \geq \sigma$. This completes the proof. \square

Obviously we say that $u \in X$ is a *critical point* if $|d_k f|(u) = 0$.

A.2 The deformation lemma

We are able now to formulate the wanted deformation property.

Theorem 15. *Let (X, d) be a metric space, and $f : X \rightarrow \mathbb{R}$ a continuous functional. Suppose that exists $\sigma \in \mathbb{R}^+$ s.t. $|d_k f|(u) \geq \sigma$ for all $u \in X$. Let $C \subset X$ be a compact subspace such that*

$$k_u(t) \leq t \quad \forall u \in X \setminus C. \quad (75)$$

Then it exists a $\tau \in \mathbb{R}^+$ and a continuous function $\mu : X \times [0, \tau] \rightarrow X$ s.t.

$$\mu(u, 0) = u \quad \forall u \in X, \quad (76)$$

$$f(\mu(u, t)) - f(u) \leq -\sigma t \quad \forall u \in X, t \in [0, \tau]. \quad (77)$$

Before proving 15, we prove two deformation lemmas for C and $X \setminus C$ analogues to lemma 4 and lemma 5. To conclude the proof we will attach the retractions found.

We recall a topological lemma by John Milnor useful for the next results.

Lemma 16 (Milnor's lemma). *Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of a paracompact space X . There is a locally finite open cover $V_{j,\lambda}$ refining $\{U_\alpha\}$ s.t. $V_{j,\lambda} \cap V_{j,\mu} = \emptyset$ if $\lambda \neq \mu$.*

Proof. For the proof we refer to [Pal66a, Lemma 2.4]. Here we report only how to construct the open cover $\{V_{i,\lambda}\}_{i,\lambda}$.

By an initial refinement we can take $\{U_\alpha\}$ locally finite. Then, let Λ_j be the set of $(j+1)$ -ples $\lambda = \{\alpha_0, \dots, \alpha_j\}$ of elements in A . Let $\{\varphi_\alpha\}_\alpha$ be a partition of unity with $\text{spt} \varphi_\alpha \subset U_\alpha$; for $\lambda \in \Lambda_j$ let

$$V_{j,\lambda} = \{x \in X \mid \varphi_\alpha > 0 \text{ if } \alpha \in \lambda \text{ and } \varphi_\gamma < \varphi_\alpha \text{ if } \alpha \in \lambda, \gamma \notin \lambda\},$$

so we have found our locally finite open cover $V_{j,\lambda}$. \square

With this lemma, we prove the deformation results.

Lemma 17 (deformation lemma for C). *Let (X, d) be a metric space, and $C \subset X$ be a compact set. Let $\sigma \in \mathbb{R}^+$ and let $f : X \rightarrow \mathbb{R}$ be a continuous function s.t.*

$$|d_k f|(u) > \sigma \quad \forall u \in C. \quad (78)$$

Then there exist $\tilde{C} \supset C$, $\tau \in \mathbb{R}^+$ and $\eta : \tilde{C} \times [0, \tau] \rightarrow X$ a continuous functional such that:

- $\eta(u, 0) = u$ for all $u \in \tilde{C}$;
- $f(\eta(u, t)) - f(u) \leq -\sigma t$ for all $u \in \tilde{C}$, $t \in [0, \tau]$.

Proof. We know by hypothesis that $|d_k f|(u) \geq \sigma$, so for every $u \in C$ there exist a $\delta_u > 0$, a continuous map $k_u : [0, \delta_u] \rightarrow \mathbb{R}^+$, $k_u(0) = 0$, and a continuous function

$$\mathcal{H}_u : B(u, \delta_u) \times [0, \delta_u] \rightarrow X$$

satisfying (73) and (74). By Milnor's Lemma we know that the open cover $\{B(u, \frac{\delta_u}{2}), u \in C\}$ admits a locally finite refinement $\{V_{j,\lambda}, j \in \mathbb{N}, \lambda \in \Lambda_j\}$ such that

$$\lambda \neq \mu \Rightarrow V_{j,\lambda} \cap V_{j,\mu} = \emptyset.$$

By compactness of C we can suppose that $\{V_{j,\lambda}\}$ be a finite family. In particular there will be an h_0 and a finite number of elements in Λ_j s.t. the family $\{V_{j,\lambda}, j = 1, \dots, h_0, \lambda \in \Lambda_j\}$ covers the whole C .

Let $\vartheta_{j,\lambda} : X \rightarrow [0, 1]$ be a family of continuous functionals with

$$\text{spt } \vartheta_{j,\lambda} \subset V_{j,\lambda},$$

$$\sum_{j=1}^{h_0} \sum_{\lambda \in \Lambda_j} \vartheta_{j,\lambda}(u) = 1.$$

For every (j, λ) let $V_{j,\lambda} \subset B(u_{j,\lambda}, \delta_{u_{j,\lambda}})$. To simplify the notations set $\delta_{j,\lambda} = \delta_{u_{j,\lambda}}$, $k_{j,\lambda} = k_{u_{j,\lambda}}$ and $\mathcal{H}_{j,\lambda} = \mathcal{H}_{u_{j,\lambda}}$. Let τ_0 be a positive real number such that $0 < \tau_0 < \min \delta_{j,\lambda}$, so every $k_{j,\lambda}$ is well defined on $[0, \tau_0]$. Let

$$k(t) = \bigvee_{j,\lambda} k_{j,\lambda}(t);$$

let τ_1 be a positive real number such that

$$\max_{t \in [0, \tau_1]} k(t) \leq \frac{1}{2} \frac{\min \delta_{j,\lambda}}{h_0 \sum_j \#\Lambda_j}. \quad (79)$$

Set $\tau = \min\{\tau_0, \tau_1\}$.

Now, called

$$\tilde{C} = \bigcup_{j,\lambda} \overline{V}_{j,\lambda},$$

we want to define a sequence of continuous map

$$\eta_h : \tilde{C} \times [0, \tau] \rightarrow X$$

such that

$$d(\eta_h(v, t), v) \leq \sum_{j=1}^h \sum_{\lambda \in \Lambda_j} k(\vartheta_{j,\lambda}(v)t), \quad (80)$$

$$f(\eta_h(v, t)) - f(v) \leq -\sigma \left(\sum_{j=1}^h \sum_{\lambda \in \Lambda_j} \vartheta_{j,\lambda}(v) \right) t. \quad (81)$$

First of all we set

$$\eta_1(v, t) = \begin{cases} \mathcal{H}_{1,\lambda}(v, \vartheta_{1,\lambda}(v)t), & \text{if } v \in \overline{V}_{1,\lambda}; \\ v, & \text{if } v \notin \bigcup_{\lambda \in \Lambda_1} V_{1,\lambda}. \end{cases}$$

Obviously η_1 satisfies (80) and (81); now we proceed by induction: assume that we have defined η_{h-1} satisfying (80) and (81). For every $v \in \overline{V}_{h,\lambda}$ we have

$$d(\eta_{h-1}(v, t), v) \leq \sum_{j=1}^{h-1} \sum_{\lambda \in \Lambda_j} k(\vartheta_{j,\lambda} t) \leq (h-1) \sum_j \# \Lambda_j \max_{t \in [0, \tau]} k(t) \leq \frac{1}{2} \delta_{h,\lambda},$$

hence $\eta_{h-1}(v, t) \in B(u_{h,\lambda}, \delta_{h,\lambda})$, so the map

$$\eta_h(v, t) = \begin{cases} \mathcal{H}_{h,\lambda}(\eta_{h-1}(v, t), \vartheta_{h,\lambda}(v)t), & \text{if } v \in \overline{V}_{h,\lambda}; \\ \eta_{h-1}(v, t), & \text{if } v \notin \bigcup_{\lambda \in \Lambda_h} V_{h,\lambda}. \end{cases}$$

is well defined and satisfies (80) and (81).

Now we set

$$\eta(u, t) = \eta_{h_0}(u, t), \quad (82)$$

so we have that $\eta : \tilde{C} \times [0, \tau] \rightarrow X$ is continuous. Furthermore

$$d(\eta(v, t), v) \leq \sum_{j=1}^h \sum_{\lambda \in \Lambda_j} k(\vartheta_{j,\lambda}(v)t) \Rightarrow \eta(0, v) = v, \quad (83)$$

$$f(\eta(v, t)) - f(v) \leq -\sigma \left(\sum_{j=1}^{h-1} \sum_{\lambda \in \Lambda_j} \vartheta_{j,\lambda}(v) \right) t = -\sigma t, \quad (84)$$

that concludes the proof \square

In this lemma we have used the compactness of C to compound the local retractions without using the property (71) of the weak slope. To find a retraction on $X \setminus C$ we must suppose that $k_u(t) \leq t$ and proceed as in Degiovanni, Marzocchi and Corvellec work [CDM93].

Lemma 18 (deformation lemma for $X \setminus C$). *Let (X, d) be a metric space; let $\sigma \in \mathbb{R}^+$ and let $f : X \rightarrow \mathbb{R}$ be a continuous function s.t.*

$$|d_k f|(u) > \sigma \quad \forall u \in X. \quad (85)$$

Suppose also that there exists a compact set $C \subset X$ such that

$$k_u(t) \leq t \quad \forall u \in X \setminus C, \quad (86)$$

where k_u is defined as in (73).

Then exist $\tau \in \mathbb{R}^+$ and $\eta : X \setminus C \times [0, \tau] \rightarrow X$ a continuous functional such that

- $\eta(u, 0) = u$ for all $u \in X \setminus C$;
- $f(\eta(u, t)) - f(u) \leq -\sigma t$ for all $u \in X \setminus C$, $t \in [0, \tau]$.

Proof. For all details see [CDM93, theorem 2.8]. We note only that the proof is quite similar to lemma 17, but for proving that η_h is well defined we must use the inequality (86) to obtain a good estimate of $d(\eta_{h-1}(v, t), v)$. \square

By lemma 17 and lemma 18 the proof of main theorem follows as usual.

Proof of theorem 15. Let $V \subset X$ be s.t. $C \subset V \subset \tilde{C}$; we can also choose V such that $B(V, \rho) \subset \tilde{C}$ for some $\rho > 0$. Set η_C and $\eta_{X \setminus C}$ the retraction found respectively in lemma 17 and 18. For the sake of simplicity we suppose that they are defined for all $t \in [0, 1]$. Let $\theta : X \rightarrow [0, 1]$ be a continuous map s.t.

$$\theta|_C \equiv 0; \quad (87)$$

$$\theta|_{X \setminus V} \equiv 1. \quad (88)$$

and let $\theta_2 = 1 - \theta_1$. Then we define a continuous map

$$\mu_1 : X \times [0, 1] \rightarrow X,$$

$$\mu_1(u, t) = \begin{cases} \eta_{X \setminus C}(u, \theta_1(u)t) & u \in X \setminus C, \\ u & \text{otherwise;} \end{cases}$$

we know that

$$f(\mu_1(u, t)) - f(u) \leq -\sigma t \theta_1(u), \quad (89)$$

and that

$$d(\mu_1(u, t), u) \leq t \theta_1(u). \quad (90)$$

Now let

$$\mu_2 = \begin{cases} \eta_C(\mu_1(u, t), \theta_2(u)t) & u \in V, \\ \mu_1(u, t) & \text{otherwise;} \end{cases}$$

we found that

$$d(\mu_1(u, t), u) \leq \theta_1(u)t \leq \rho \quad \text{if } t \leq \rho,$$

so μ_2 is well defined on $X \times [0, \rho]$.

Obviously we have that

$$\mu_2(u, 0) = 0 \quad \forall u \in X; \quad (91)$$

furthermore, if $u \in X \setminus V$ we have that

$$f(\mu_2(u, t)) - f(u) = f(\mu_1(u, t)) - f(u) \leq -\sigma t, \quad (92)$$

and that, if $u \in V$, then

$$\begin{aligned} f(\mu_2(u, t)) - f(u) &= f(\eta_C(\mu_1(u, t), \theta_2(u)t)) - f(u) = \\ &= f(\eta_C(\mu_1(u, t), \theta_2(u)t)) - f(\mu_1(u, t)) + f(\mu_1(u, t)) - f(u) \leq \\ &\quad -\sigma \theta_2(u)t - \sigma \theta_1(u)t = -\sigma t. \end{aligned} \quad (93)$$

So, we set $\tau = \rho$ and $\mu = \mu_2$ and we conclude the proof. \square

We provide a final remark: we observe that if there exist a compact set $C \subset X$, then we are allowed to weaken the standard definition of weak slope. to compound the local retraction explicitly.

In $X \setminus C$ we must recover condition (71) adding the hypothesis (75) of theorem 15: in a non compact set this estimate makes possible a continuous composition of local retractions.

References

- [CD95] Annamaria Canino and Marco Degiovanni, *Nonsmooth critical point theory and quasilinear elliptic equations*, Topological methods in differential equations and inclusions (Montreal, PQ, 1994), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 472, Kluwer Acad. Publ., Dordrecht, 1995, pp. 1–50.
- [CDM93] Jean-Nöel Corvellec, Marco Degiovanni, and Marco Marzocchi, *Deformation properties for continuous functionals and critical point theory*, Topological Methods in Nonlinear Analysis **1** (1993), 151–171.
- [Deg97] Marco Degiovanni, *Nonsmooth critical point theory and applications*, Proceedings of the Second World Congress of Nonlinear Analysts, Part 1 (Athens, 1996), vol. 30, 1997, pp. 89–99.
- [DM94] Marco Degiovanni and Marco Marzocchi, *A critical point theory for nonsmooth functionals*, Annali di Matematica Pura ed Applicata (IV) **167** (1994), 73–100.
- [DM99] Marco Degiovanni and Laura Morbini, *Closed geodesics with Lipschitz obstacle*, J. Math. Anal. Appl. **233** (1999), no. 2, 767–789.
- [FH91] Edward Richard Fadell and Sufian Yunis Husseini, *Category of loop spaces of open subsets in Euclidean space*, Nonlinear Anal. **17** (1991), no. 12, 1153–1161.
- [Ghi04] Marco G. Ghimenti, *Geodesics in conical manifolds*, Phd thesis, Università degli studi di Pisa, Dipartimento di Matematica, 2004.
- [Kli78] Wilhelm Klingenberg, *Lectures on closed geodesics*, Springer-Verlag, Berlin, 1978, Grundlehren der Mathematischen Wissenschaften, Vol. 230.
- [LS34] Lazar Aronovich Lusternik and Lev Genrichovic Schnirelmann, *Méthodes topologiques dans les problèmes variationnels*, Hermann, Paris, 1934.
- [MM02] Marco Marzocchi and Laura Morbini, *Periodic solutions of Lagrangian systems with Lipschitz obstacle*, Nonlinear Anal. **49** (2002), no. 2, Ser. A: Theory Methods, 177–195.
- [MS83] Antonio Marino and Donato Scolozzi, *Geodetiche con ostacolo*, Boll. Un. Mat. Ital. B (6) **2** (1983), no. 1, 1–31, (italian).
- [Nas56] John Nash, *The imbedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), no. 1, 20–63.
- [Pal66a] Richard S. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology **5** (1966), 1–16.
- [Pal66b] ———, *Lusternik-Schnirelman theory on Banach manifolds*, Topology **5** (1966), 115–132.

- [Rab74] Paul H. Rabinowitz, *Variational methods for nonlinear eigenvalue problems*, Eigenvalues of non-linear problems (Giovanni Prodi, ed.), Centro internazionale matematico estivo (C.I.M.E.), Edizioni Cremonese, Roma, June 1974.